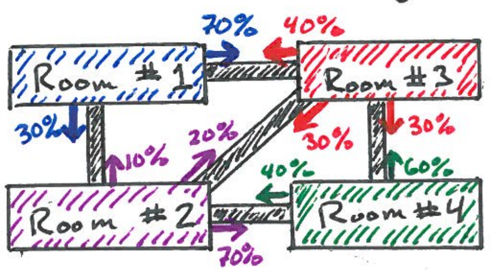
A (discrete time) Markov process is a system consisting of a series of "states". Every time period there is movement between the states according to some given "transition probabilities".

Two basic types of question can be investigated:

Question 1: Given a starting state, what is the probability of being in another target state after some number of time periods?

Question 2: Compute the "long-term" probability of being in some state after a large number of time periods.

EX: Four rooms are connected by hallways as drawn. Every hour people change rooms.



From Room #1,
30% go to #2, 70% to #3

From Room #2,
10% go to #1, 20% to #3,
70% to #4

From Room #3,
40% go to #1, 30% to #2, 30% to #4

From Room #4,
40% go to #2, 60% to #3

The "states" in this system are the different rooms where a person could be

States = {Room #1, Room #2, Room #3, Room #4}

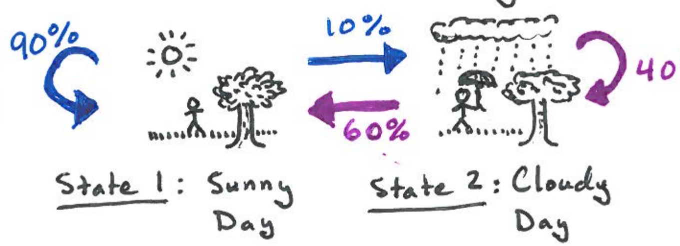
The "transition probabilities" are the proportion of people who go to each room

e.g. Prob(from #1 to #3) = .7
Prob(from #1 to #4) = 0

EX: Weather in Cyprus is either rainy or sunny.

If today is sunny then the probability that tomorrow is also sunny is 90%.

If today is rainy then the probability that tomorrow is also rainy is 40%.



Transition probabilities naturally form a matrix — the "transition matrix" K.

- Columns of K give probabilities for the different target states coming from a fixed state
- Rows of K give probabilities for the different source states going to a fixed state

EX: In the example with rooms, the transition matrix is

$$K = \begin{array}{c} \begin{array}{cccc} \text{From Room \#1} & \text{From Room \#2} & \text{From Room \#3} & \text{From Room \#4} \end{array} \\ \begin{bmatrix} 0 & .1 & .4 & 0 \\ .3 & 0 & .3 & .4 \\ .7 & .2 & 0 & .6 \\ 0 & .7 & .3 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{To Room \#1} \\ \text{To Room \#2} \\ \text{To Room \#3} \\ \text{To Room \#4} \end{array}$$

(Note: In this example the diagonal is all 0, because no one is allowed to stay in their room.)

EX: The transition matrix for the weather example is given by

$$K = \begin{array}{c} \begin{array}{cc} \text{Today Sunny} & \text{Today Rainy} \end{array} \\ \begin{bmatrix} .9 & .6 \\ .1 & .4 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Tomorrow Sunny} \\ \text{Tomorrow Rainy} \end{array}$$

The elements of transition matrices have two special properties:

(1) all entries are numbers between 0 & 1

(2) the sum of all entries in a column is 1

A matrix satisfying these properties (even if it isn't a transition matrix) is called a "Markov" or "stochastic" matrix.

Note: It does not matter which order you use for the states in the transition matrix as long as you use the same order for rows and columns.

→ For example, for the weather transition matrix we could also have used

$$K = \begin{array}{c} \overbrace{\begin{array}{cc} \text{Rainy} & \text{Sunny} \end{array}}^{\text{Today}} \\ \begin{bmatrix} .4 & .1 \\ .6 & .9 \end{bmatrix} \end{array} \left.\begin{array}{l} \\ \text{Rainy} \\ \text{Sunny} \end{array}\right\} \text{Tomorrow}$$

The current state of a system is written as a vector. The "<u>state vector</u>" can mean different things depending on the context.

<u>EX</u>: In the rooms example, the state vector

$$\mathbf{v} = \begin{bmatrix} 100 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{l} \leftarrow \text{\# of people in room 1} \\ \leftarrow \text{\# of people in room 2} \\ \leftarrow \text{\# of people in room 3} \\ \end{array}$$

would mean

100 people in room 1
0 people in room 2
0 people in room 3
0 people in room 4.

The product $K\mathbf{v}$ of the transition matrix and state vector gives

$$\begin{bmatrix} 0 & .1 & .4 & 0 \\ .3 & 0 & .3 & .4 \\ .7 & .2 & 0 & .6 \\ 0 & .7 & .3 & 0 \end{bmatrix} \begin{bmatrix} 100 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 30 \\ 70 \\ 0 \end{bmatrix} \begin{array}{l} \leftarrow \text{in room 1} \\ \leftarrow \text{in room 2} \\ \leftarrow \text{in room 3} \\ \leftarrow \text{in room 4} \end{array}$$

the number of people in each room after 1 hour.

Multiplying by K again gives

$$\begin{bmatrix} 0 & .1 & .4 & 0 \\ .3 & 0 & .3 & .4 \\ .7 & .2 & 0 & .6 \\ 0 & .7 & .3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 30 \\ 70 \\ 0 \end{bmatrix} = \begin{bmatrix} 31 \\ 21 \\ 6 \\ 42 \end{bmatrix} \begin{array}{l} \leftarrow \text{in room 1} \\ \leftarrow \text{in room 2} \\ \leftarrow \text{in room 3} \\ \leftarrow \text{in room 4} \end{array}$$

the number of people in each room after 2 hours.

More generally, $K^n\mathbf{v}$ will give the number of people in each room after $\underline{\underline{n}}$ hours.

<u>EX</u>: In the weather example, the state vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{array}{l} \leftarrow \text{sunny} \\ \leftarrow \text{rainy} \end{array}$$

corresponds to today being sunny.

(100% sunny & 0% rainy)

The product $K\mathbf{v}$ of the transition matrix and state vector gives

$$\begin{bmatrix} .9 & .6 \\ .1 & .4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .9 \\ .1 \end{bmatrix} \begin{array}{l} \leftarrow \text{sunny} \\ \leftarrow \text{rainy} \end{array}$$

the probability of sun & rain tomorrow.

Multiplying by K again gives

$$\begin{bmatrix} .9 & .6 \\ .1 & .4 \end{bmatrix} \begin{bmatrix} .9 \\ .1 \end{bmatrix} = .9 \begin{bmatrix} .9 \\ .1 \end{bmatrix} + .1 \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .87 \\ .13 \end{bmatrix}$$

the probability of sun & rain the day after tomorrow.

More generally, $K^n v$ will give the probability of sun & rain after $\underline{n}$ days.

We now have a solution to the first question — at least, using either a lot of work or a calculator... After writing the transition matrix and state vector for a Markov system, the probability of being in different states after n time steps is

$$\underline{\underline{K^n v}}$$

MatLab Break!

Some example computations in MatLab:
We can enter matrices into MatLab using spaces between entries and semi-colons between rows.

```
>> K = [.9 .6 ; .1 .4]
```

The state vector for sun today is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

```
>> v = [1 ; 0]
```

The probability of sun & rain tomorrow is $K \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

```
>> K * v
```

Multiplying again gives probabilities for the day after tomorrow.

```
>> K * ans
```

And the day after that...

```
>> K * ans
```

(you can use the "up" arrow to repeat the previous command...)

To get probabilities for next week, it may be faster to use matrix powers

```
>> K^7 * v
```

Something interesting happens... once the number of days (the matrix power) gets large, the probabilities stabilize!

```
>> K^100 * v
>> K^200 * v
>> K * ans
>> K * ans
```

These all give the same answer!
"After enough days have passed, one more day doesn't really make a difference."

Even more interesting: you reach the same stable solution no matter what your initial state is!

```
>> K^100 * [1;0]
>> K^100 * [0;1]
>> K^100 * [.5;.5]
```

"Partly rainy" day.

These all give the same answer!
"After enough days have passed, no one remembers what the weather was originally like."

The same thing happens to the Markov system with people in 4 rooms... after enough time passes, the distribution of people in rooms stabilizes.

```
>> K = [0 .1 .4 0
        .3 0 .3 .4
        .7 .2 0 .6
        0 .7 .3 0]
```

You can also enter matrices by typing each row on its own line.

```
>> K*[100;0;0;0]
>> K * ans
>> K * ans
```

repeat a few more times...

```
>> K^100 * [100;0;0;0]
>> K^200 * [100;0;0;0]
>> K * ans
```

Also it doesn't matter where the 100 people start... after enough time passes the number of people in each room will end up the same!

```
>> K^40 *[100; 0; 0; 0]    <- Begin in room #1
>> K^40 *[0; 100; 0; 0]    <- Begin in room #2
>> K^40 *[0; 0; 100; 0]    <- Begin in room #3
>> K^40 *[0; 0; 0; 100]    <- Begin in room #4
```

This is called the "stable state" for the Markov system — after more and more transitions occur, all states are pulled towards the stable state.

→ This happens in almost all Markov systems.

The stable state is useful because it is very easy to find — even without a computer!

$$\boxed{\text{If } v \text{ is the stable state, then } K\mathbf{v} = \mathbf{v}}$$

This is called a "1-eigenvector" of K

Note: If $\mathbf{v}$ is a stable state then so are all multiples of $\mathbf{v}$

$$K(c\mathbf{v}) = c(K\mathbf{v}) = c\mathbf{v}$$

EX Find the stable state probability for the weather system

$$\begin{bmatrix} .9 & .6 \\ .1 & .4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{cases} .9a + .6b = a \\ .1a + .4b = b \end{cases}$$

$Kv = v$

$(K - 1_{\text{on diag}})v = 0$

$$\begin{cases} (.9-1)a + .6b = 0 \\ .1a + (.4-1)b = 0 \end{cases}$$

$$\begin{cases} -.1a + .6b = 0 \\ .1a - .6b = 0 \end{cases} \rightarrow a = 6b \qquad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} b$$

To get steady state probability choose b so that the state vector sums to 1

$$b = \frac{1}{6+1}$$

$$\boxed{\text{steady state probability} = \begin{bmatrix} 6/7 \\ 1/7 \end{bmatrix}}$$

6/7 of days sunny & 1/7 of days rainy

Looking at the previous computation, there is an obvious simplification we can make to our method. To solve $Kv = v$ we must convert to $\left(K - \begin{smallmatrix}1 \text{ on} \\ \text{diagonal}\end{smallmatrix}\right) v = 0$

$$\boxed{\text{To find the stable state ("1-eigenvector") solve } \left(K - \begin{smallmatrix}1 \text{ on} \\ \text{diagonal}\end{smallmatrix}\right) v = 0}$$

(**Note:** If you are using a computer, it may be faster to just compute K^(big #) * (random vector))

Note: Solving by hand becomes terribly difficult very quickly. Solving the rooms example would require dividing by a 4x4 matrix full of fractions... I will not do that in class.

EX: There are two rooms. Every hour 1/4 of the people in room 1 go to room 2 and 1/3 of the people in room 2 go to room 1. If 140 people begin in room 1, find the long term distribution.

$$K = \begin{array}{c|c} & \overbrace{\#1 \quad\quad \#2}^{\text{From}} \\ & \begin{bmatrix} 3/4 & 1/3 \\ 1/4 & 2/3 \end{bmatrix} \begin{matrix} \#1 \\ \#2 \end{matrix}\Big\}\text{To} \end{array}$$

Missing probabilities are $1 - 1/4 = 3/4$, $1 - 1/3 = 2/3$

$$\left(K - \begin{smallmatrix}1 \text{ on} \\ \text{diag}\end{smallmatrix}\right) = \begin{bmatrix} -1/4 & 1/3 \\ 1/4 & -1/3 \end{bmatrix}$$

$$\begin{bmatrix} -1/4 & 1/3 \\ 1/4 & -1/3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \underline{\underline{3a = 4b}}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4/3 \\ 1 \end{bmatrix} b$$

To get probability vector $\begin{bmatrix} 4/3 \\ 1 \end{bmatrix} b \rightsquigarrow \begin{bmatrix} 4 \\ 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 4/7 \\ 3/7 \end{bmatrix}$

Choose b so that sum is $\underline{\underline{140}}$

$$\left(4/3 + 1\right) b = 140 \Rightarrow b = 60$$

$$\begin{bmatrix} 4/3 \\ 1 \end{bmatrix} 60 = \begin{bmatrix} 80 \\ 60 \end{bmatrix}$$

$$\boxed{\begin{matrix} 80 \text{ in room } 1 \\ 60 \text{ in room } 2 \end{matrix}}$$

Check $\begin{bmatrix} 3/4 & 1/3 \\ 1/4 & 2/3 \end{bmatrix}\begin{bmatrix} 80 \\ 60 \end{bmatrix} = \underbrace{80\begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}}_{\text{from room 1}} + \underbrace{60\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}}_{\text{from room 2}} = \begin{bmatrix} 60 \\ 20 \end{bmatrix} + \begin{bmatrix} 20 \\ 40 \end{bmatrix}$ $\underline{\underline{\text{OK.}}}$